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Intermittent behaviour in non-linear-Hamiltonian systems far from equilibrium

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Abstract. We show that the approach to equilibrium in non-linear-Hamiltonian systems exhibits strongly non-gaussian features (intermittency). The intensity of the phenomenon grows fainter until it vanishes near equilibrium. We present a heuristic interpretation of such behaviour confirmed by numerical simulations on the system described by a one-dimensional classical ϕ^4 field theory.

1. Introduction

Intermittency, one of the most interesting phenomena of non-linear dissipative systems (e.g. turbulence), is not yet well understood either experimentally or theoretically (Monin and Yaglom 1975 (§ 25), Manneville and Pomeau 1980).

The physical quantities that describe the process (e.g. a velocity gradient) exhibit long, quiet periods alternating with short periods of strong activity (intermittent bursts).

Intermittency can be interpreted in the theory of turbulence as an irregular transfer of energy—a peculiar non-linear feature—from each scale of motion to the next shorter one (Frisch *et al* 1978). Such a flow of energy occurs on every scale concerning the motion up to the dissipative scale. This situation may appear as a stationary state from the statistical point of view. Turbulence indeed exhibits intermittency both when approaching the stationary state and when it has been reached. The latter case is explained by the dissipative nature of the turbulent system whose steady state is maintained by means of a constant external energy pumping into large scales.

To our knowledge, few attempts have been made to explore the presence of intermittency in non-dissipative systems: only Kraichnan (1967, 1975) and Frisch and Morf (1981) tried to connect intermittency with the general properties of the non-linear equations that describe a system, disregarding its dissipative nature.

The main reason for this deficiency is probably that discussions on turbulence often introduce the intermittency as being related to a concentration of dissipated energy in small, irregular regions (Corrsin 1962, Saffman 1968). The quoted interpretation has only recently been adopted generally.

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In conservative systems we could conclude that, since there is no energy dissipation, no phenomenon like intermittency can exist. However, Kraichnan (1975) first advanced the hypothesis that intermittency could subsist in conservative systems far from equilibrium by the following argument.

He considered systems described by the Euler equation of energy conservation: $\frac{1}{2}\Sigma_k |\boldsymbol{v}(\boldsymbol{k})|^2 = \text{constant}$ (where $\boldsymbol{v}(\boldsymbol{k})$ is the Fourier transform of the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$), together with the usual arguments of equilibrium statistical mechanics (e.g. the maximum entropy principle), yields the probability distribution

$$P(\{\boldsymbol{v}(\boldsymbol{k})\}) \sim \exp\left(-\beta \sum_{\boldsymbol{k}} |\boldsymbol{v}(\boldsymbol{k})|^2\right).$$

This result was confirmed by numerical simulations (Orszag and Patterson 1972). The equilibrium probability distribution of $\{\boldsymbol{v}(\boldsymbol{k})\}$ is a gaussian and therefore the presence of intermittency should be excluded.

Let us now consider the way the system approaches equilibrium. Initially, all the energy is confined into the large scales; then the system evolves towards an equipartitioned final state by virtue of the non-linear terms of the interaction which couple the motion scales among themselves.

Kraichnan stresses the point that such an energy 'cascade' mechanism from large to small scales is universal for both dissipative and non-dissipative systems. For non-dissipative systems, however, the energy flow is damped provided small scales (large k) are completely filled (i.e. energy equipartition is attained).

In this paper we attempt to verify the above predictions on a non-linear-Hamiltonian system—a classical ϕ^4 field theory—studied previously by Fucito *et al* (1982). In § 2 we sketch our analytic treatment of the system and exhibit its intermittent behaviour during its approach to equilibrium from numerical simulations. In § 3 we present a heuristic interpretation of the results obtained.

2. The model and numerical results

Fucito *et al* (1982) and Marchesoni and Sparpaglione (1982) studied the time evolution of the system described by the motion equation

$$\ddot{\phi} = \partial_x^2 \phi - m^2 \phi - g\phi^3 \tag{2.1}$$

where $\phi(x, t)$ with $x \in [-\frac{1}{2}L, \frac{1}{2}L]$ is a one-dimensional scalar field subject to periodic boundary conditions.

Our numerical simulations are carried out by discretising equation (2.1) upon a string of length $L = 1$ composed of N equidistant points. The field $\phi(x, t)$ is then represented by the variables $\phi_i(t) = \phi(i\Delta x - \frac{1}{2}L, t)$ with $i = 1, N$, where $\Delta x = 1/N$ and the boundary condition is rewritten as $\phi_{N+1} = \phi_1$.

We analyse the situation in which the energy system is initially concentrated in the smallest wavenumber modes. We realise such a configuration by choosing, for example, the initial conditions as

$$\phi_i(0) = \frac{1}{(2\pi)^{1/2}} \sum_{m=1}^{N/2} (a_m(0) \cos[2\pi(i-1)m/N] + b_m(0) \sin[2\pi(i-1)m/N]) \tag{2.2}$$

$$\dot{\phi}_i(0) = 0$$

with

$$a_1(0)=a_2(0)=A \qquad a_{i>4}(0)=0 \qquad \forall i=1,N$$
$$a_3(0)=a_4(0)=\tfrac{1}{2}A \qquad b_i(0)=0$$

when $N=128$ is adopted.

The method of numerical integration we use is called the central difference method (Benettin *et al* 1980):

$$\phi_i(t+\Delta t)=2\phi_i(t)-\phi_i(t-\Delta t)+(\Delta t)^2F_i(\{\phi_i(t)\}) \tag{2.3}$$

where

$$F_i(\{\phi_i\})=(\phi_{i+1}-2\phi_i+\phi_{i-1})/(\Delta x)^2-m^2\phi_i-g\phi_i^3$$

and the initial condition $\dot{\phi}(0)$ corresponds to $\phi_i(-\Delta t)=\phi_i(0)$. The integration procedure is stable in a large range of the adopted values for Δt (5×10^{-3}–10^{-5}) and N (64–1024) and conserves the energy to within 0.1%.

Fucito *et al* (1982) studied the time behaviour of the spectrum both numerically and theoretically:

$$W_n(t)=|a_n(t)|^2+|b_n(t)|^2. \tag{2.4}$$

The following results have been verified for several values of t (0.1–100), A (5×10^{-3}–10) and g (0.2–10):

$$W_n(t)\sim\exp(-S(t)k_n) \qquad k_n=2\pi n/L \tag{2.5}$$

with

$$S(t)=\begin{cases}-\ln(g^{1/2}At) & \text{'short time regime'}\\ (gA^2\ln t)^{-1/2} & \text{'intermediate time regime'}\end{cases} \tag{2.6}$$

where the time scale that separates the two regimes depends on the initial conditions and the strength of the non-linear coupling ($(gA^2)^{-1/2}$).

The results of equations (2.5) and (2.6) have been obtained theoretically by generalising an idea due to Frisch and Morf (1981). Let us assume that the analytic continuation $\phi(z,t)$ of the field $\phi(x,t)$ in the complex plane $z=(x,y)$ is an analytic but non-entire function: equation (2.5) follows immediately with $S(t)=2y_s(t)$, where $y_s(t)$ is the imaginary part of the nearest singularity of $\phi(z,t)$ to the real axis (for the derivation of equation (2.6) see Fucito *et al* (1982)).

A consequence of equation (2.6) is that the system approaches equilibrium with a logarithmic dependence on t, so that the non-equilibrium spectrum may persist for extremely long times and may be mistaken for a stationary state if the observation time is not sufficiently long.

Persistence of the trend described by equation (2.5) for very long times is crucial for our heuristic interpretation of the intermittent behaviour of the system which we will expound. In order to characterise such an effect quantitatively, we exploit a quantity usually introduced in this kind of problem (Monin and Yaglom 1975 (§ 25), Frisch *et al* 1978): the kurtosis F. For a random variable $\eta(x)$, F is defined as

$$F=\frac{\langle(\eta-\langle\eta\rangle)^4\rangle}{\langle(\eta-\langle\eta\rangle)^2\rangle^2}. \tag{2.7}$$

If $\eta(x)$ is a gaussian variable then $F=3$; larger values of F mean that the probability distribution of η allows strong fluctuations in comparison with the mean values $\langle\eta\rangle$.

As we noticed in § 1, we expect that large deviations from gaussian characterise the energy propagation from large to short scales. This effect is enhanced if we choose to observe quantities more appropriate to the dynamics of the large wavenumber modes for which the intermittent features should be more relevant.

We investigated two different choices. The former consists in computing the kurtosis of the spatial derivatives of the field at the time t:

$$\xi^{(n)}(x,t)=\partial_x^n\phi(x,t)$$

according to the definition

$$F_\xi^{(n)}(t)=\frac{1}{\Delta N}\int_{t-\Delta/2}^{t+\Delta/2}\mathrm{d}\tau\sum_{i=1}^{N}(\xi_i^{(n)}(\tau))^4\Big/\left(\frac{1}{\Delta N}\int_{t-\Delta/2}^{t+\Delta/2}\mathrm{d}\tau\sum_{i=1}^{N}(\xi_i^{(n)}(\tau))^2\right)^2. \tag{2.8}$$

In figures 1(a, b, c) we show the time dependence of $F_\xi^{(n)}(t)$ for different values of n and A. We note immediately the following features.

(i) The time behaviour of F is characterised by high peaks which appear initially at regular intervals of time. At larger times F achieves values close to the characteristic value of the gaussian processes. These different behaviours correspond to the short and intermediate time regimes, respectively, as can be seen by comparison with the time dependence of $S(t)$ drawn in figure 1.

(ii) For small values of A the initial structure remains longer according to the energy dependence of the short time regime that is known to persist for smaller energies.

(iii) The values of F increase with the order of the field derivatives n, but the structure of the peaks is preserved. This feature confirms the idea that intermittency characterises the energy flow from large to short scales since higher-order derivatives favour the role of the short scales much more.

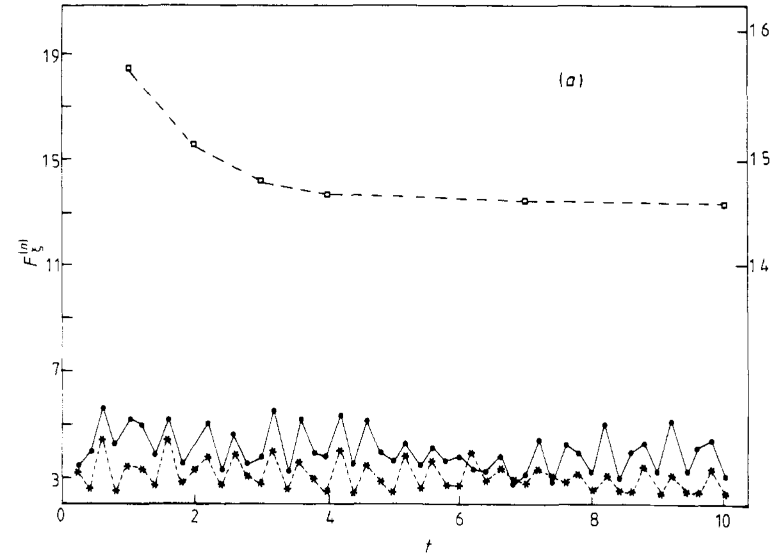

Figure 1.

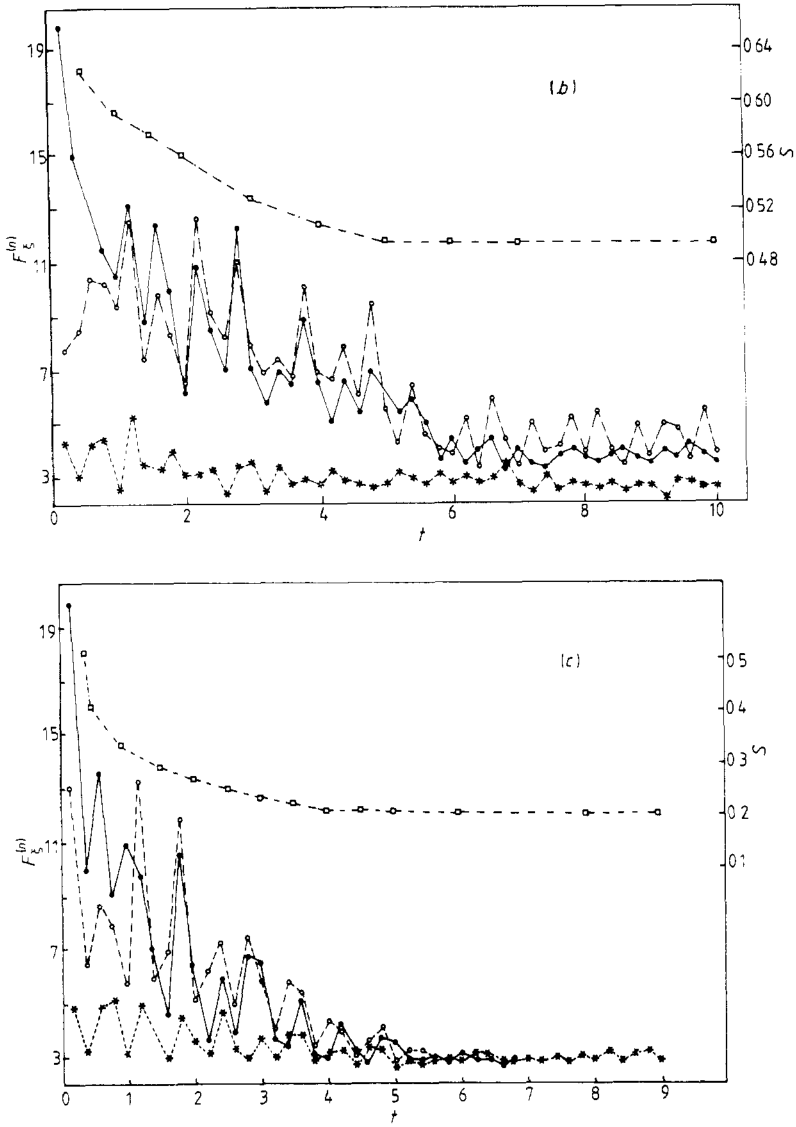

Figure 1. $F_\zeta^{(n)}$ and S against time; $g=5$, $\Delta=0.2$, $N=128$. ●——●, $n=6$; ○——○, $n=3$; $*$ – – $*$, $n=1$; □– –□ S. (*a*) $A=1$, (*b*) $A=5$, (*c*) $A=10$.

We have obtained the same results studying the variable introduced by applying numerically a high band-pass filter on the field $\phi(x, t)$. In other words, we cut away the Fourier components with smaller wavenumbers ($k > n$) and computed the kurtosis of the reconstructed field:

$$\zeta^{(n)}(x, t) = \frac{1}{(2\pi)^{1/2}} \sum_{m \geq n} (a_m(t) \cos(2\pi(m-1)x/N) + b_m(t) \sin(2\pi(m-1)x/N)).$$

In order to have a check of the systematic effects due to the numerical method of integration at the same time, we introduce the following definition of kurtosis in this case:

$$F_\zeta^{(n)} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\Delta}\int_{t-\Delta/2}^{t+\Delta/2} \mathrm{d}\tau\, \zeta_i^{(n)}(\tau)\right)^4 \Bigg/ \left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\Delta}\int_{t-\Delta/2}^{t+\Delta/2} \mathrm{d}\tau\, \zeta_i^{(n)}(\tau)\right)^2\right]^2. \tag{2.9}$$

We notice that the definitions (2.8) and (2.9) of kurtosis differ because the average in F_ξ is essentially a time average while that in F_ζ is a spatial one.

In figures 2(a, b, c) we show the results obtained for the same values of A and g adopted in figure 1.

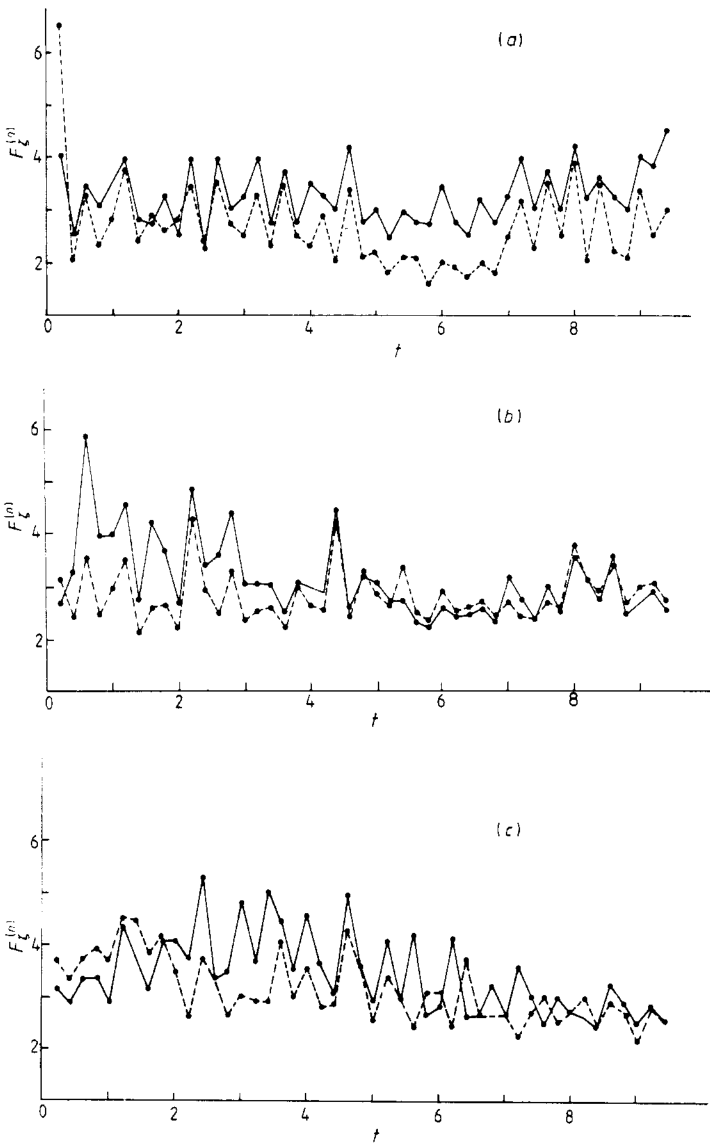

Figure 2. $F_\zeta^{(n)}$ against time; $g = 5$, $\Delta = 0.2$, $N = 128$. ●——●, $n = 8$; ●- - - -●, $n = 6$. (a) $A = 1$, (b) $A = 5$, (c) $A = 10$.

3. Interpretation of the numerical results

In this section we propose a heuristic interpretation of the results presented above. Firstly we have to justify the large values of the kurtosis appearing in the numerical simulations of § 2.

To this aim we develop the idea of Frisch and Morf (1981). If $\zeta^{(n)}(x, t)$ is the reconstructed field through a high band-pass filter with $k > k_n$ $(k_n = 2\pi n/N)$, we can assert that in the short time regime only one pole in the complex z plane dominates the intermittent behaviour of the system (Fucito *et al* 1982, Frisch and Morf 1981):

$$\zeta^{(n)}(x, t) = \frac{(x - x_s)\cos[k_n(x - x_s)] + y_s \sin[k_n(x - x_s)]}{(x - x_s)^2 + y_s^2} \exp(-y_s k_n). \quad (3.1)$$

Equation (3.1) describes an x-modulated intermittent burst with wavenumber k_n. The modulation envelope is centred at the real part x_s of the singularity and decreases in inverse proportion to the separation from the centre. There is an overall amplitude factor which favours the singularities close to the real axis. The symmetry $\phi(-x, t) = \phi(x, t)$ of equation (2.1) and of the initial conditions implies the contemporary existence of another burst centred at $-x_s$.

If we now substitute $\zeta^{(n)}$ of equation (3.1) into equation (2.9) for $F_\zeta^{(n)}$, we obtain a function which increases with n, i.e. with the frequency threshold of the filter. Moreover, its time behaviour is governed by the time dependence of $y_s(t)$. In particular a peak structure is apparent in both figures 1 and 2, and can be understood by assuming that the position of the closest pole to the real axis $y_s(t)$ oscillates.

Let us consider only the short time regime for which the above assumption is immediately justified. We demonstrated that in such an approximation

$$y_s(t) \sim -\ln(g^{1/2}At)/k_0$$

(see equation (2.6)), so that the pole falls from infinity towards the real axis with a characteristic time proportional to $(gA^2)^{-1/2}$. Such a trend should be interpreted as a sort of average over small time intervals (Fucito *et al* 1982) close to which the instantaneous position of the pole fluctuates. Indeed several contributions neglected in the naive approach yielding equation (2.6) also concur to determine the actual trajectory of the pole $y_s(t)$.

We notice that in equation (3.1) $y_s(t)$ appears as the argument of an exponential: this implies that even very small variations of y_s can induce large variations of the kurtosis values and give rise to the irregular structures of F_ξ and F_ζ shown in figures 1 and 2.

The long time behaviour of the kurtosis can also be interpreted in the frame of our model. Fucito *et al* (1982) showed that the intermediate time regime (in figure 1 it corresponds to the flat end of $y_s(t)$) is characterised by the contemporary presence of many singularities close to the real axis. This means that, whatever their trajectories may be, the distance of the nearest pole to the real axis $y_s(t)$ changes slightly with time and that intermittent bursts of different intensities arise and overlap in such a way that an intermediate quasi-ordered condition with small kurtosis values is restored.

This explanation of the time behaviour of F_ξ and F_ζ is in qualitative agreement with the observed decrease of the activity interval of the kurtosis as A increases. Indeed as the intermediate time regime is achieved, several singularities grouped near the real axis stabilise the values of the kurtosis close to the quasi-equilibrium values.

Nevertheless figure 3 shows that some kind of intermittency survives in the intermediate time regime. This fact is in agreement with the Kraichnan (1967) argument about the additional presence of intermittency for observables with distributions like equation (2.5). In addition, the spectrum of the field $\phi(x, t)$ retains the form (2.5) till the large wavenumber modes are almost filled by the energy flow. We can conclude that an intermittent behaviour should persist throughout the approach to equilibrium (energy equipartition).

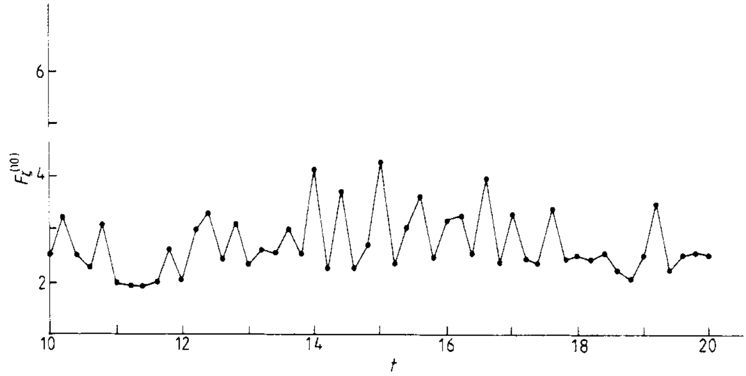

Figure 3. $F_\zeta^{(10)}$ against time; $g = 5$, $\Delta = 0.2$, $N = 128$.

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